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THE

AMERICAN

MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

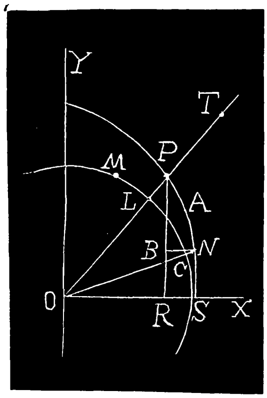
VOL. XIV. JUNE-JULY, 1907. NOS. 6-7.

ON DIVIDING AN ANGLE INTO PARTS HAVING THE RATIOS OF ANY GIVEN STRAIGHT LINES.*

By REV. R. D. CARMICHAEL, Presbyterian College, Anniston, Alabama.

The solution of this problem will be here effected by aid of the locus of the polar equation $\rho\sin\theta=m\theta$. In a similar manner it may be carried out by means of the locus of $\rho\cos\theta=m\theta$. We shall take the special case $m=1$ of the first equation, thus giving $\rho\sin\theta=\theta$.

It is desirable to have a method of constructing the curve by continuous motion. We proceed in the following manner: Construct a material circle with center O and radius unity, as in the figure, and let it be fixed to the plane of the paper. Let O be the origin and OX the polar axis. Let P be any point on the curve, and let OPT be a straight bar pivoted to the paper at O and free to move without carrying the circle with it. Draw PR perpendicular to OX; now we have $OP=\rho$, $\angle POR=\theta$, and $PR=\rho\sin\theta$. Then from the equation of the curve, $PR=\theta$. That is, PR is numerically equal in length to the arc LCS. Now fasten one end of a cord at some point Q in the circumference of the circle and pass it any convenient number of times around the circle in the direction QLS. On the last round let the free end of the cord pass through a small ring fastened to OT at L. Then let it pass around a roller at T and back through another small ring at P, with its free end taking the direction PR perpendicular to OX at R, the extremity of the cord being at R. Take now another chord of length $LT+PT$, with its ends attached to the rings at L and P, the cord also passing around the roller at T. As the line OT moves, let the roller at T so slide as to keep the last chord tightly stretched. (This may be accomplished by having T attached to a spring.) The purpose of this chord is to make the distance $LT+PT$ remain canstant as OT moves. If a pencil is also fastened



*Presented to the American Mathematical Society, February 23, 1907.

to the ring at P and passed through a slit in OT, it will furnish a means of marking on the paper the path of P. If OT moves toward OY so that L comes to any point as m, a portion of the first cord equal to the arc LM will be set free. Now if P, R, and T should also move so that they remain in the positions defined above, the pencil at P will trace on the paper an arc of the locus, as may be very easily proven. PR and OX may be kept at right angles by means of a sliding right triangle with one leg in RX and the other in RP. The proper relative positions of L, P, and T are easily found when OT coincides in direction with OX. Hence the method given is a complete solution of the problem of the construction of the curve by continuous motion.

We shall now apply this locus to the problem of dividing an angle into parts having the ratios of any given straight lines. Let the given angle take the position POS in the figure. If the line PR is divided in the given ratio (as it is always easy to do) and through the points of section lines are drawn parallel to OX intersecting the arc PNS in the points A, N, etc.; then if the lines OA, OM, etc., are drawn, they will divide the angle POS into the required parts.

PROOF. Let B be one of the points of section on PR as defined above. Draw BN parallel to OX. Draw ON. It is evidently sufficient to our demonstration to prove that

$$\angle PON : \angle NOS :: PB : BR;$$

for then we can evidently cut off from one side of the angle one of the proportional parts; from a side of this another; and so on. This clearly gives the same points A, N, etc., as above. A perpendicular let fall from N on OX is equal to BR. Hence from the method of constructing the locus, we know that

the line BR is numerically equal to the arc CS.

Also, the line PR is numerically equal to the arc LS.

Hence, $$\text{arc } LS : \text{arc } CS :: PR : BR.$$

Then, by division of ratios,

$$\text{arc } LC : \text{arc } CS :: PB : BR.$$

Hence, $$\angle PON : \angle NOS :: PB : BR.$$

Therefore, the problem is completely solved by means of the locus of $\rho \sin \theta = \theta$.

REMARK. A special case of this is the problem of dividing an angle into n equal parts. It will be noticed that in the present case it is not necessary to construct different curves for different values of n, but that one

curve gives the solution for every case. Attention is also called to the fact that by this method an angle may be divided in any ratio or ratios in which a straight line may be divided. It is therefore as easily separated into parts having an incommensurable ratio as into any other, provided that that incommensurable ratio can be expressed by means of straight lines.

NOTE ON THE VOLUME OF A TETRAHEDRON IN TERMS OF THE COORDINATES OF THE VERTICES.

By DR. L. E. DICKSON.

1. Quite a variety of propositions of solid analytic geometry are needed for the usual derivation of the volume of a tetrahedron (cf. C. Smith, p. 24). If, as in the present note, we give an elementary proof making use merely of the concept of coordinates, we are in a position to apply the result to derive* easily several of the initial propositions in solid analytics, *e. g.*, that the equation of any plane is of the first degree, and conversely.

The plan of the proof (§ 3) is entirely obvious. The only novelty lies in a certain device which yields the result without computation. This device will first be illustrated in deriving the area of a triangle (§ 2).

2. Let the vertices of a triangle $\triangle$ taken in counter-clockwise order be (x_1, y_1), (x_2, y_2), (x_3, y_3). Then $\triangle$ can be expressed in terms of three right trapezoids with parallel sides y_i. The area of a right trapezoid with parallel sides y_1 and y_2, and base b, is $\frac{1}{2}b(y_1+y_2)$, being half of the rectangle of height y_1+y_2 and base b. Hence

$$2\triangle=(x_1-x_2)(y_1+y_2)+(x_2-x_3)(y_2+y_3)+(x_3-x_1)(y_3+y_1).$$

The device consists in setting $s=y_1+y_2+y_3$. Then

$$2\triangle=(x_1-x_2)(s-y_3)+(x_2-x_3)(s-y_1)+(x_3-x_1)(s-y_2).$$

Since each x occurs once positively and once negatively, the terms in s evidently cancel. The remaining terms give the expansion, according to the second column, of

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

*For plane analytics, this plan is followed in the chapter on graphic algebra in the writer's *College Algebra* (John Wiley and Sons).